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SOLUTION OF NONLINEAR PROBLEM OF COMBINED HEAT TRANSFER BY DIFFERENT APPROXIMATIONS OF HEAT TRANSFER EQUATIONS ON DIFFERENT-SCALE GRIDS

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An efficient method for solution of the nonlinear problem of combined heat transfer based on the simultaneous use of zonal and finite-difference approximations of heat transfer equations on different-scale grids within the framework of a unified inexplicit iteration scheme is proposed. The efficiency and accuracy of the approach proposed are demonstrated on a test example.

Simultaneous heat transfer by means of radiation, heat conduction, and mass transport is a distinctive feature of problems of the combined heat transfer. Another peculiarity of applied problems of the type consists in the substantial differences in the spatial scales of the regions where calculations are carried out: heating of stocks, radiation pipes in the working space of furnaces, etc. To date, a number of efficient methods for the approximation of particular heat transfer equations have been developed. The finite-difference method is frequently used for the solution of heat conduction equations [1]. The zonal method is successfully used for calculation of the radiative heat transfer [2, 3], and equations of energy conservation and hydrodynamics are efficiently solved by the control volume method [3]. The finite-element method has received wide recognition in CAD systems [4]. In problems where the processes under investigation are induced by physical processes of different natures, one must to employ hybrid methods of calculations. A difference in spatial scales leads to the necessity of the application of grids with substantially differing scales, since the generation of homogeneous fine grids is not always admissible or justified, as, e.g., in the case of the calculation of the radiative heat transfer in the virtually isothermal working space of a furnace.

There exist two main approaches to the solution of problems of combined and conjugate heat transfer. The first approach is realized by employing the same method of approximation of various heat transfer equations on a common grid. The desired temperatures are defined by solution of a single global system of equations. In [1], the method of large abrupt changes of velocity at a liquid–solid interface has been used to solve the problem of conjugate heat transfer. However, a strong abrupt change in the coefficients of the equations reduces the convergence rate of iteration schemes.

Conversely, various methods of approximation of heat transfer equations on different grids are used in the second method. The desired temperatures are determined by successive solution of the system of heat transfer equations for corresponding processes and geometric regions with fixed source terms, temperatures, and heat fluxes on the boundaries of the regions. In the conventional approach to the solution of the problem of conjugate heat transfer, the external problem of radiation-convection transfer is solved in the first stage to determine the heat fluxes on the boundary. Calculated heat fluxes play the role of boundary conditions for the internal problem of conductive heat transfer, from which the temperature values for the external problem are refined [2]. Relaxation parameters should be used or other precautions should be taken to provide convergence in this approach [2]. In the zone–node method, zone-averaged temperatures calculated by solution of nonlinear equations are used for calculation of the radiation source terms in energy conservation equations written for a fine finite-difference grid [3]. Temperatures and convective and conductive heat fluxes of control volumes are used to form convective-conductive coefficients for the zone grid. The need to form coefficient matrices for both grids and absence of a diagonal predominance in the convective matrix, which results in slow convergence of iteration schemes [2], are

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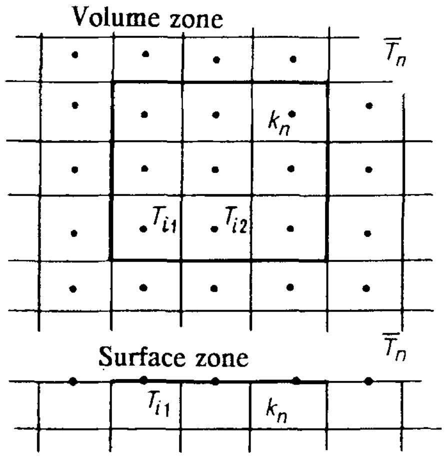

Fig. 1. Matching of zonal and finite-difference grids.

drawbacks of the method. In [5], in order to combine the external and internal heat transfer problems for a thermally thin body, the temperatures at the nodes of control volumes were expressed in terms of zone temperatures. In this case, the matrix of coefficients in the equation of the heat balance loses its diagonal predominance.

In [6], an algorithm for solution of a problem the conjugate heat transfer employing the finite-difference method for the external problem and the boundary-element methods for the internal one, has been proposed. The finite-difference and finite-element grids were coincident on the interface of the regions. Temperature values are calculated successively for each region. To ensure convergence, special values of relaxation parameters should be chosen.

In the present work we consider an efficient method for the solution of a problem of combined heat transfer which is free of the above-mentioned drawbacks. The method employs simultaneously different-scale grids for the zone and finite-difference approximation methods.

Basic Equations. The calculation of thermal modes of energy technological objects based on the approximation of integral heat transfer equations can be reduced to the solution of nonlinear equations written for each discrete element of the physical system [2]:

$$\rho_i c_i V_i \frac{\partial T_i}{\partial \tau} + \sum_{j=1}^{N} R_{ij} T_j^4 + \sum_{j=1}^{N} D_{ij} T_j + S_i = 0 , \quad i = 1, ..., N . \tag{1}$$

Here T_j are the absolute temperatures of the discrete elements and S_i are terms connected with external sources and boundary conditions. Coefficients in Eq. (1) can be obtained on the basis of various methods of approximation of heat transfer equations.

The Newton–Rafson method [4] is the most efficient approach to the solution of nonlinear equation (1) for both the time-inexplicit scheme and stationary formulation, and for a system of nonlinear equations presented in vector form

$$\mathrm{f}(\mathrm{X}) = 0 ,$$

can be written as

$$\mathrm{J}(\mathrm{X}^l)\, \Delta \mathrm{X}^{l+1} = - \mathrm{f}(\mathrm{X}^l) , \tag{2.a}$$

$$\mathrm{X}^{l+1} = \mathrm{X}^l + \Delta \mathrm{X}^{l+1} , \tag{2.b}$$

where $\mathrm{J}(\mathrm{X}^l)$ is a Jacobian of the vector function $\mathrm{f}(\mathrm{X})$: X^l is a vector of independent variables, and l is the number of iterations over the system of nonlinear equations. As a rule, four to six iterations provide a solution of Eq. (1) with accuracy acceptable in engineering calculations.

By assuming that the coefficients of Eq. (1) depend only on the temperature of the discrete element for which the heat balance equation is written, we present the elements of the Jacobian in the following form:

$$J_{ij}(T_j^J) = \delta_{ij}\frac{V_i}{\Delta\tau}\rho_i c_i + 4R_{ij}(T_j^J)^3 + D_{ij}\frac{\partial R_{ij}}{\partial T_j}(T_j^J)^4 + \frac{\partial D_{ij}}{\partial T_j}T_j^J + \delta_{ij}\frac{V_i}{\Delta\tau}\frac{\partial(\rho_i c_i)}{\partial T_j}T_j^J.$$

The index connected with the time layer is omitted for simplicity.

In the case of large-dimension problems, the system of linear equations (2a) is efficiently solved by an iteration method. Methods of conjugate directions with preconditioning [7-10] provide good results in this case.

Figure 1 presents portions of regions where heat transfer is realized by radiation, conduction, and convection. The radiative heat transfer was calculated by the zone method on a sparse zonal grid (shown by thick solid lines in Fig. 1). The approximation of heat fluxes due to the other two processes is carried out by the finite-difference method on a fine grid. The conditions of coincidence of the grids are as follows: the volume zone includes an integer number of control volumes, and the surface zone covers faces of an integer number of control volumes. Zones and control volumes are numbered by indices m, n and i, j, respectively. Inasmuch as there is no substantial difference in the coincidence of surface and volume zones with the grid of control volumes, we will consider in what follows only the case of coincidence of the volume zone. Let us denote the set of numbers of control volumes entering into the n-th zone as $N_n = \{i_1, i_2, ..., i_{k_n}\}$. For simplicity, we will consider a stationary formulation of the problem. By using assumptions of the resolvent zone method, one can present the radiation heat flux of the n-th zone as follows [2]:

$$Q_n^R = \sum_{m=1}^{N_z} R_{nm}\bar{T}_m^4. \tag{3}$$

By approximating the conductive and convective heat fluxes of the control volume entering into the n-th zone using the finite-difference method, we generally obtain [1]

$$Q_i^D = \sum_{j=1}^{N} D_{ij}T_j, \quad i \in N_n. \tag{4}$$

Let us write the heat balance equation for the n-th zone

$$Q_n^R + \sum_{i\in N_n} Q_i^D + S_n + \sum_{i\in N_n} S_i = 0, \tag{5}$$

where S_n are source terms and boundary conditions in the problem of radiative heat transfer, and S_i are source terms in the problem of conductive-convective transfer not included in S_n. Substituting (3) and (4) into (5), we obtain

$$\sum_{m=1}^{N_z} R_{nm}\bar{T}_m^4 + \sum_{i\in N_n}\sum_{j=1}^{N} D_{ij}T_j + S_n + \sum_{i\in N_n} S_i = 0. \tag{6}$$

Then we transform the second term in expression (6):

$$\sum_{i\in N_n}\sum_{j=1}^{N} D_{ij}T_j = \sum_{j=1}^{N}\sum_{i\in N_n} D_{ij}T_j = \sum_{j=1}^{N} D'_{nj}T_j = \sum_{j\in N_n} D'_{nj}T_j + \sum_{j=1,\, j\notin N_n}^{N} D'_{nj}T_j. \tag{7}$$

When a first-order finite-difference scheme is used for diffusion heat fluxes and a counterflow scheme is used for convective heat fluxes, only the coefficients for the elements situated on the boundary of the n-th zone enter into expression (7). The first and second terms on the right-hand side of (7) include the temperatures of the boundary control volumes of the n-th zone and those of the control volumes of the immediate vicinity of the n-th zone, respectively.

Equation (6) is a heat balance equation for volume zone. Let us assume that the temperatures of the nodes of the control volumes entering into the n-th zone are equal. Then, taking into account that the temperature T_j in the first term of (7) is replaced by $\overline{T}_n$, we have

$$\sum_{i\in N_n} D'_{nj} T_j + \sum_{j=1,\, j\notin N_n}^{N} D'_{nj} T_j = \overline{T}_n \sum_{j\in N_n} D'_{nj} + \sum_{j=1,\, j\notin N_n}^{N} D'_{nj} T_j . \tag{8}$$

If the temperatures of the nodes of the control volumes from N_n are not equal, equation (8), speaking strictly, is not satisfied in the nonlinear problem.

By introducing the following notation

$$S'_n = S_n + \sum_{i\in N_n} S_i .$$

we present Eq. (6) with regard for (7) and (8) as follows:

$$\sum_{m=1}^{N_z} R_{nm} \overline{T}_m^4 + \overline{T}_n \sum_{j\in N_n}^{N} D'_{ij} + \sum_{j=1,\, j\notin N_n}^{N} D'_{ij} T_j + S'_n = 0 , \quad n = 1, ..., N_z . \tag{9}$$

Let us write the heat balance for the i-th control volume entering into the n-th zone:

$$Q_i^D = Q_i^R + S_i = 0 , \quad i \in N_n . \tag{10}$$

Let us assume that the volume density of the resulting radiation heat flux is uniform within the selected zone. Then one can write

$$Q_i^R = \frac{V_i}{V_n} Q_n^R = \frac{V_i}{V_n} \sum_{m=1}^{N_z} R_{nm} \overline{T}_m^4 + \frac{V_i}{V_n} S_n ,$$

where S_n is the source term of the n-th zone different from sources S_i. Then

$$Q_i^R = \sum_{m=1}^{N_z} R'_{nm} \overline{T}_m^4 + \frac{V_i}{V_n} S_n + R'_{nn} \overline{T}_n^4 + \sum_{m=1,\, m\neq n}^{N_z} R'_{nm} \overline{T}_m^4 + \frac{V_i}{V_n} S_n .$$

Now, again, assuming that the temperatures of the nodes of all control volumes entering into a single zone are equal, we replace temperature $\overline{T}_n$ by the temperature of the control volume T_i. Then equation (10) with regard for the substitution $S'_i = S_i + S_n V_i / V_n$ is as follows

$$\sum_{m=1,\, m\neq n}^{N_z} R'_{nm} \overline{T}_m^4 + R'_{nn} T_i^4 + \sum_{j=1}^{N} D'_{ij} T_j + S'_i = 0 , \quad i \in N_n . \tag{11}$$

Modified equations (9) and (11) are included in a global system of coupled nonlinear heat balance equations (1), which is then solved by the Newton–Rafson method simultaneously for the temperatures of the control volumes and the zone temperatures, which are independent variables. To solve linearized equation (2a), any iteration method of conjugate directions with preconditioning would be appropriate. The GMRES and TFQMR methods [8, 9] have been found to work well in this case.

Test Problem. As a test of the scheme proposed, we consider the problem of evaluation of heat fluxes and temperature fields in a water-cooled roller of a sectional furnace which frequently arises in practice. A typical configuration of a roller situated within a furnace section is presented in Fig. 2a. Under certain assumptions, the

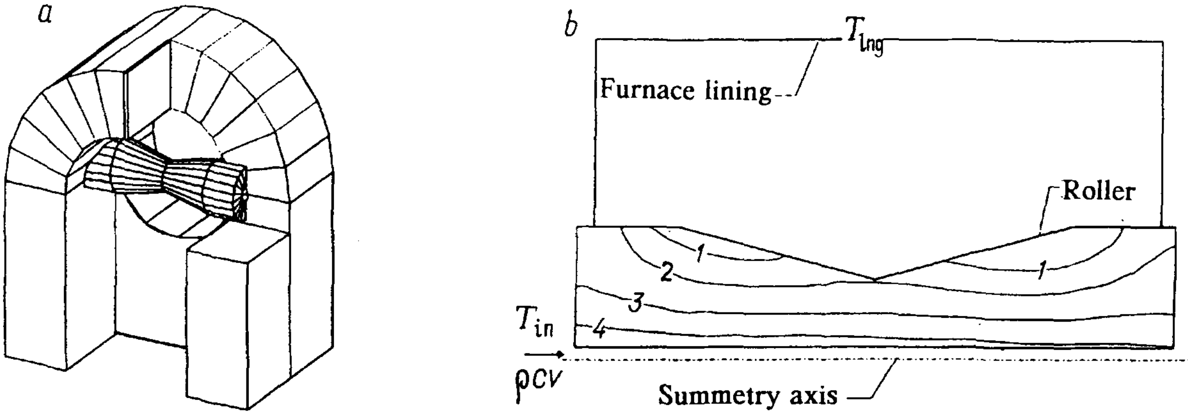

Fig. 2. Typical configuration of roller in working space of furnace (a) and corresponding axisymmetric model (b); curves, isolines of the field of relative temperatures: 1) $\Theta = 0.5$; 2) 0.4; 3) 0.25; 4) 0.1; Sn = 0.01; Bi = 2; St = 0.2.

problem of calculation of the temperature field in the roller can be reduced to an axisymmetric problem of conjugate heat transfer with the geometry shown in Fig. 2b. The roller is heated due to the radiation from the furnace lining with temperature T_{lng}, and is cooled by a liquid entering its internal channel with initial temperature $T_{in} = 0$ K. The convective heat transfer in the working space of the furnace was not taken into account. The solution of the problem is a function of the three dimensionless Stanton, Biot, and Stark numbers:

$$\mathrm{Sn} = \frac{\alpha}{\rho c v}, \quad \mathrm{Bi} = \frac{\alpha \bar{h}}{\lambda}, \quad \mathrm{St} = \frac{\sigma T_{lng}^3 \bar{h}}{\lambda}.$$

The roller was divided into control volumes. The conductive heat transfer was approximated by the finite-difference method. The internal channel of the roller was also divided into control volumes along the radial direction. The heat transfer due to the motion of the cooling liquid was approximated by counterflow differences. The radiative heat transfer was calculated by the zone method. On a portion of the roller surface situated within the working space of the furnace, surface areas were selected each of which covered an integer number of neighboring control volumes. The free surface of the roller not exposed to the working space of the furnace and not belonging to the internal channel was considered to be adiabatic.

To estimate an accuracy of the coupled scheme proposed, we obtained a provisionally exact solution of the problem of conjugate heat transfer. Heat fluxes due to radiation, heat conduction, and convection were approximated on a fine grid. A single surface zone was put in correspondence with each face of the control volume situated in the working space of the furnace. The temperature of this surface zone was identified with that of the corresponding control volume. Unknown temperatures were calculated by solution nonlinear heat-balance equation (1) written for the stationary case. Isolines of the calculated relative temperature $\Theta = T(x, y)/T_{lng}$ are presented in Fig. 2b.

Figure 3a presents a dependence of the maximum relative error of the calculation of the heat flux on the gas–solid interface for the coupled scheme as a function of the value of the parameter St. The error was calculated with respect to heat fluxes of the provisionally exact solution Q^*

$$\delta_Q = \max |(Q - Q^*)/Q^*| \cdot 100\% .$$

Three cases were considered when a single surface zone covered faces of 4, 8, and 40 neighboring control volumes (CV) situated on the roller surface.

Figure 3b presents a dependence of the maximum relative error with respect to the temperature and heat flux as a function of the number of control volumes n whose external faces were included in the surface zone. The error was calculated with respect to the temperature of the provisionally exact solution $T^*(x, y)$

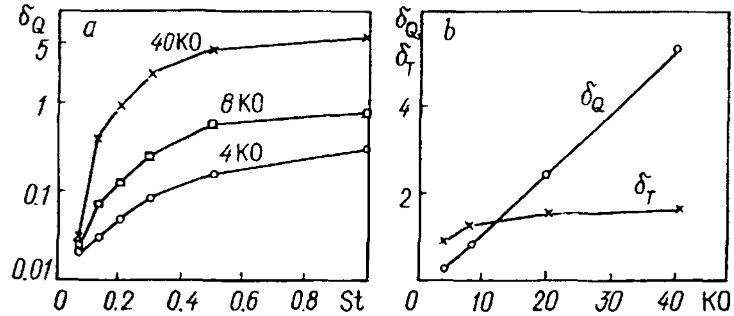

Fig. 3. Dependences of δ_Q (%) on St (a); δ_Q and δ_T (%) on the number of control volumes covered by a single zone (b): a) Sn = 0.01; Bi = 2; b) Sn = 0.01; Bi = 2; St = 1.

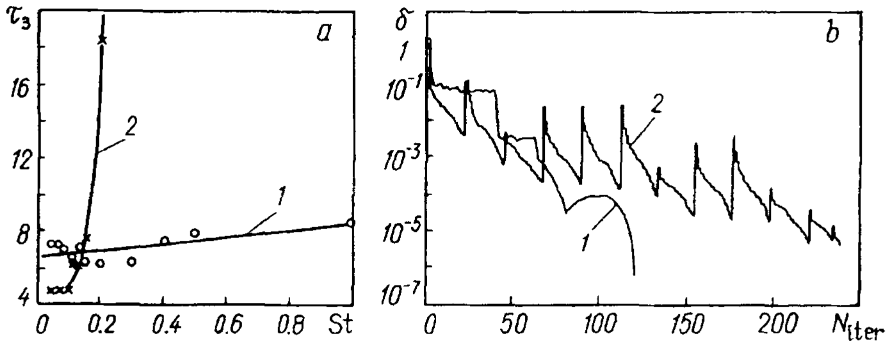

Fig. 4. Comparison of convergence rates of two computational schemes under the condition that a single zone covers faces of four control volumes: a) time necessary to reduce the discrepancy of Eq. (1) by a factor of 10^6 as a function of St; b) dependence of relative discrepancy of Eq. (1) on the number of iterations over the linearized system: 1) coupled scheme; 2) fixed-boundary-condition scheme; a single zone covers faces of four control volumes: a) Sn = 0.01; Bi = 2; b) Sn = 0.01; Bi = 2; St = 0.2.

$$\delta_T = \max \left| (T(x, y) - T^*(x, y))/T^*(x, y) \right| \cdot 100\% .$$

Figure 4a presents results of a comparison of the convergence rate for the two methods of solution of the test problem. Each surface zone covered faces of four control volumes. Heat balance equations were written for the stationary formulation. The temperature distribution was calculated by the following two methods. In the first method, a conventional algorithm was used in which the external and internal heat transfer problems were solved successuvely with use of the temperature and heat flux conjugation conditions on the boundary. In what follows, we will refer to it as the fixed boundary-condition scheme.

In the second case, the coupled computational scheme (1), (2), (9), (11) was used. The system of linear equations in both cases was solved by one of two variants of the quasiminimum discrepancy method (TFQMR) with the ILU preconditioning [7, 8]. The solution was considered to be found if the length of the discrepancy vector decreased by a factor of 10^6. The computational experiment was carried out on an IBM-compatible personal computer equipped with a 130-MHz Pentium processor.

Figure 4b shows the dependence of the length of the relative discrepancy vector $\delta = |f(T)|/(\sigma F T_{\text{lng}}^4)$ on the iteration number N_{iter} for the two above schemes.

A computational experiment has shown the acceptable accuracy and high convergence rate of the coupled scheme proposed in the present work. In the coupled scheme, the error of the calculation of heat fluxes on the boundary depends on the temperature jumps between nodes of control volumes entering into a single zone. An increase in the jumps is observed with an increasing number of control volumes covered by a single zone, and for small St numbers (Fig. 3). When St > 0.5, the temperature jump on the roller surface decreases in the working space. Therefore, the relative error of calculations weakly depends on the St number.

Despite the fact that the method proposed does not provide a gain in the convergence rate at small St numbers, it guarantees the solution of problems where conventional schemes break down or require the special choice of relaxation parameters. In this case, the convergence rate of the coupled scheme weakly depends on the parameters of the problem (Fig. 4a). In the given example, the gain in time required for solution of the problem observed for the fixed boundary conditions scheme for St < 0.1 compared to the coupled scheme stems from the fact that the former one is not connected with the solution of a system of nonlinear equations, since the lining temperature is considered to be given.

Conclusion. The coupled scheme for the solution of the nonlinear problem of combined heat transfer proposed in the present work has a series of advantages. In addition to the possibility of employing the advantages of the zonal and finite-difference methods and different-scale grids, which is important in the solution of practical problems, this scheme does not involve the use of relaxation parameters or other special means to provide convergence of the iteration procedures.

The inclusion of equations written for various grids in a global system of equations is based on a purely algebraic transformation of the coefficients obtained for the original grids. As a result, there is no need to form coefficient matrices separately for both sparse and fine grids. Although an increase in the number of nonzero elements of the global matrix of the linearized system is observed with respect to original matrices written separately for different grids, the filling of the matrix remains substantially lower than in multigrid methods, where, to calculate radiative heat transfer on a fine grid, one must solve a system with a virtually completely filled matrix [11].

We have developed a method for solution of a nonlinear problem of combined heat transfer which can be efficiently used in systems for computer-aided design of thermal modes of actual energy-technology objects by power engineers who are not necessarily experts in computational methods.

NOTATION

ρ, density; c, specific heat; V_i, volume of the i-th control volume; V_n, volume of the n-th zone; v, velocity of motion of cooling liquid; σ, the Stefan–Boltzmann constant; τ, time; F, area of lining surface; Sn, Stanton number; Bi, Biot number; St, Stark number; T_j, absolute temperature of nodes of control volumes; $\overline{T}_m$, average zone temperatures; R_{ij}, coefficients approximating radiative heat transfer; D_{ij}, coefficients approximating conductive and convective heat transfer; Q^R, resulting radiation heat transfer; Q^D, resulting heat flux due to heat conduction and convection; δ_{ij}, the Kronecker symbol; α, coefficient of convective heat transfer from fluid to roller; $\overline{h}$, average thickness of roller; λ, heat conductivity of roller material; δ_Q, relative error of evaluation of resulting heat fluxes; δ_T, relative error of evaluation of temperatures; δ, length of relative discrepancy vector in test problem; τ_p, time required for solution of test problem with specified accuracy; Θ, relative temperature; T_{lng}, lining temperature; T_{in}, temperature of cooling liquid at entrance to internal channel of roller; N, total number of discrete elements in system; N_z, number of zones; N_n, number of indices of control volumes entering into the n-th zone; N_{iter}, number of iterations in solution of test problem.

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